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MINIMUM WEIGHT BARS FOR GIVEN LOWER BOUNDS ON EULER BUCKLING LOAD AND FREQUENCY OF LONGITUDINAL VIBRATION

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Abstract—The necessary optimality conditions are developed for the problem of minimizing the mass of a structural member subject to design constraints on two fundamental eigenvalues, namely frequency of longitudinal vibration and Euler buckling load. The regions of the design space, in which each of the constraints is active, are delineated. An effective numerical solution procedure is derived and solutions are obtained for a wide range of the design variables for beams of both solid cross-section and sandwich construction. The optimal designs are compared with a prismatic beam satisfying the design constraints.

1. INTRODUCTION

Considerable attention has been given to eigenvalue problems in search of optimal designs for elastic members. For example the problem of maximizing the buckling load of an elastic bar of given mass was posed by Lagrange[1] but the correct solution for a bar with simply supported ends was not found until almost a century later by Clausen[2] and still later, but independently, by Keller[3].

The optimal designs of columns with other homogeneous boundary conditions were obtained by various authors[4–7]. The problem of maximizing the fundamental frequency of transverse vibration of a simply support beam was solved by Niordson[8], other authors[9–12] considered beams and plates with various boundary conditions. The problem of the optimum design of a bar with respect to longitudinal vibration was solved by Turner[13].

An extensive review of the optimization of structural elements with respect to structural eigenvalues is given by Olhoff[14].

Seyranian[15, 16] has discussed the problem of optimizing a beam subjected to multiple constraints including constraints on eigenvalues. No solutions were presented; rather he proposes a "quasioptimal" solution method in which a solution of the optimization problem subject to a single constraint is scaled so as to be made admissible to the problem with multiple constraints and used as a "quasioptimal solution" to that problem. Karihaloo and Parbery have examined the problems of optimizing a beam with dual constraints on buckling load and fundamental frequency of transverse vibration[17], and with constraints on the fundamental frequencies of longitudinal and transverse vibrations[18]. A problem on the optimization of a beam subjected to three eigenvalue constraints has been discussed although no solutions were obtained[19]. Błachut [20] has dealt with the problem of optimizing a column with tip mass under stability and transverse vibration constraints. The problem considered by Błachut is of a different type to those considered in Refs [17, 18] and the present paper, inasmuch as in the former work the load actions are considered to occur simultaneously whereas in the latter they are considered to occur at different times in the life of the member.

In the work reported in this paper an algorithm has been developed for solving the problem of minimizing the weight of a beam subjected to constraints on the fundamental frequency of longitudinal vibration and on the buckling load. Numerical results have been found, in the case of a cantilevered beam with a non-structural mass at the tip, for a range of problem parameters.

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2. FORMULATION OF THE PROBLEM

Consider an elastic bar of length L, cross-sectional area $A(x^*)$, axial coordinate x^*, with principal moment $I(x^*)$ normal to the plane of bending and made of material with Young's modulus E and density ρ. It is expected that the bar will be subjected to, at different times in its design life, longitudinal vibration and axial static loads. It is desired to find the variation of cross-sectional area along the member length which minimizes the weight (volume) of the member for given lower bounds on the two eigenvalues mentioned above.

It is assumed that the moment of inertia I and the area A of the cross-section are related by

$$I = cA^n \tag{1}$$

where c and n are constants characteristic of the cross-sectional form.

For example $n \simeq 1$ for a beam of sandwich construction or an I beam of constant depth, $n = 2$ for geometrically similar cross-sections of variable dimensions, and $n = 3$ for a rectangular beam of constant width but variable depth.

The differential equation describing the two types of behaviour are, in non-dimensional form (refer to Fig. 1)

for longitudinal vibrations

$$(\alpha_1 u_x)_x + \beta\alpha_1 u = 0; \tag{2}$$

for buckling

$$\alpha_1^n w_{xx} + P(w - w(1)) = 0. \tag{3}$$

Equations (2) and (3) must be accompanied by boundary conditions reflecting the support conditions of the member and of any non-structural mass.

(i) For a cantilever with added mass at the tip the boundary conditions are

$$u(0) = 0 \tag{4}$$

$$(\alpha_1 u_x)|_{x=1} = M\beta u|_{x=1} \tag{5}$$

$$w(0) = w_x(0) = 0. \tag{6}$$

(ii) Boundary conditions for a pin-ended beam

$$u(0) = u(1) = 0 \tag{7}$$

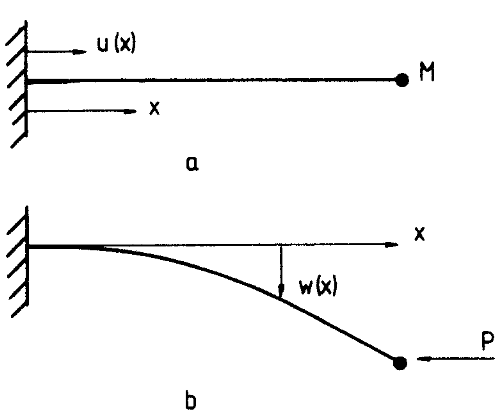

Fig. 1. Beam with added mass at tip subjected to (a) longitudinal vibration and (b) Euler buckling.

$$w(0) = w(1) = 0. \tag{8}$$

Equations (2)–(8) have been put into non-dimensional form by letting

$$\alpha_1 = A/L^2, \quad x = x^*/L, \quad u = u^*/L, \quad w = w^*/L,$$
$$\beta = \Omega^2 \rho L^2/E, \quad P = P^*/EcL^{2(n-1)}, \quad M = M^*/\rho L^3$$

where $u^*(x^*)$ is the axial displacement in longitudinal vibration, $w^*(x^*)$ the deflection in buckling, Ω the fundamental frequency of longitudinal vibration, P^* the Euler buckling load and M^* the added mass. The asterisks, where used, indicate dimensional quantities.

We seek the function $\alpha_1(x)$ on [0, 1] which minimizes the weight of the beam subject to given lower bounds on the fundamental eigenvalues β and P. Forming the Rayleigh quotients for each of the eigenvalues, and assuming the density ρ to be constant, one may reduce the optimization problem under consideration to

$$\min_{\alpha_1(x)} V = \int_0^1 \alpha_1(x)\, \mathrm{d}x \tag{9}$$

subject to

$$\beta = \frac{\int_0^1 \alpha_1 u_x^2\, \mathrm{d}x}{\int_0^1 \alpha_1 u^2\, \mathrm{d}x + \varepsilon M} \geqslant \beta_0 \tag{10}$$

$$P = \frac{\int_0^1 \alpha_1^n w_{xx}^2\, \mathrm{d}x}{\int_0^1 w_x^2\, \mathrm{d}x} \geqslant P_0 \tag{11}$$

where β_0, P_0 are prescribed positive constants; $\varepsilon = 0$ for a simply supported beam and $\varepsilon = 1$ for a cantilevered beam; and where the eigenfunctions have been normalized by making the maximum deflections equal to unity. Such normalization is possible because the eigenfunctions are determined only as to shape and not to magnitude.

In order to derive the necessary optimality condition for the optimization problem above, the following auxiliary functional is formed:

$$\Pi = \int_0^1 \alpha_1\, \mathrm{d}x + \mu\left[(\beta_0 + r^2)\left(\int_0^1 \alpha_1 u^2\, \mathrm{d}x + \varepsilon M\right) - \int_0^1 \alpha_1 u_x^2\, \mathrm{d}x\right]$$
$$+ \xi\left[(P_0 + t^2)\int_0^1 w_x^2\, \mathrm{d}x - \int_0^1 \alpha_1^n w_{xx}^2\, \mathrm{d}x\right] \tag{12}$$

where μ and ξ are (constant) Lagrange multipliers and r^2 and t^2 are positive slack variables. We require that Π be stationary with respect to α_1, r and t.

Considering first the variations in Π due to r and t

$$\delta\Pi_r = 2\mu r\left(\int_0^1 \alpha_1 u^2\, \mathrm{d}x + \varepsilon M\right)\delta r = 0 \tag{13}$$

$$\delta\Pi_t = 2\xi t\left(\int_0^1 w_x^2\, \mathrm{d}x\right)\delta t = 0 \tag{14}$$

noting that the terms in parentheses are positive and that the variations in r and t are arbitrary, one obtains, respectively

$$\mu = 0 \quad \text{or} \quad r = 0 \tag{15}$$

$$\xi = 0 \quad \text{or} \quad t = 0. \tag{16}$$

If any of the Lagrange multipliers vanish it means that the corresponding constraint is inactive, i.e. the inequality sign applies in the corresponding constraint, eqn (10) or (11).

Assuming, for the present, that all constraints are active, that is $r = t = 0$, one obtains from the stationarity of Π with respect to α_1

$$\int_0^1 [1+\mu(\beta_0 u^2 - u_x^2) - \xi n \alpha_1^{n-1} w_{xx}^2]\,\delta\alpha_1 \,\mathrm{d}x = 0. \tag{17}$$

Recognizing that $\delta\alpha_1$ is an arbitrary function, one obtains the following optimality condition

$$\mu(u_x^2 - \beta_0 u^2) + \xi n \alpha_1^{n-1} w_{xx}^2 = 1. \tag{18}$$

Equation (18) together with eqns (15) and (16) represents the necessary optimality condition for the problem which must be solved together with the relevant equations, eqns (2) and (3), with appropriate boundary conditions (4)–(6) or (7) and (8) and eigenvalue constraints (10) and (11).

3. SOLUTION OF THE PROBLEM

The solution will now be considered in more detail for the particular example of the cantilever with an added mass at the tip.

It is convenient to normalize α_1 by M. Then the relevant equations, eqns (2)–(6), (10) and (11), become, respectively

$$(\alpha u_x)_x + \beta \alpha u = 0 \tag{19}$$

$$\alpha^n w_{xx} + \frac{P}{M^n}(w - w(1)) = 0 \tag{20}$$

$$u(0) = 0 \tag{21}$$

$$(\alpha u_x)|_{x=1} = \beta u|_{x=1} \tag{22}$$

$$w(0) = w_x(0) = 0 \tag{23}$$

$$\beta = \frac{\displaystyle\int_0^1 \alpha u_x^2 \,\mathrm{d}x}{\displaystyle\int_0^1 \alpha u^2 \,\mathrm{d}x + 1} \geqslant \beta_0 \tag{24}$$

$$\frac{P}{M^n} = \frac{\displaystyle\int_0^1 \alpha^n w_{xx}^2 \,\mathrm{d}x}{\displaystyle\int_0^1 w_x^2 \,\mathrm{d}x} \geqslant \frac{P_0}{M^n} \tag{25}$$

where $\alpha = \alpha_1/M$.

The optimality condition (18) can be rewritten as

$$\mu u_x^2 + \xi \alpha^{n-1} w_{xx}^2 = 1 + \beta_0 \mu u^2 \tag{26}$$

where nM^{n-1} have been included in ξ.

3.1. *Special cases*

Before discussing the solution in general it is necessary to consider the special cases suggested by eqns (15) and (16) and delineate the regions of the design space where either one or both of the constraints, eqns (24) and (25), are active.

When $\xi = 0$, $\mu > 0$ in the optimality condition (26), the problem is equivalent to that of minimizing the volume subject to the constraint on the fundamental frequency of longitudinal vibration alone. The solution to this problem was found by Turner[13] and is described by the following equations:

$$u(x) = \sinh (Cx)/\sinh (C) \tag{27}$$

$$\alpha(x) = C \sinh C \cosh C/\cosh^2 (Cx) \tag{28}$$

$$\text{Volume} = \sinh^2 C \tag{29}$$

where

$$C = \sqrt{\beta_0}.$$

When the beam optimally designed for longitudinal vibrations is used as a column, the critical buckling load may be found from eqns (20), (23) and (25) where α is given by eqn (28). A standard iterative method was used to evaluate the values of P/M^n corresponding to various given values of β_0. The results are shown graphically in Figs 2–4.

The second special case is when $\mu = 0$, $\xi > 0$. This corresponds to the case of an optimally designed column, which when used in the longitudinal vibration mode, will have a fundamental frequency greater than that specified. Calculation of the frequency is not straightforward because the optimal column[3–6] exhibits $\alpha(1) = 0$; so that, in view of boundary condition (22), u_x must exhibit a singularity at $x = 1$. As is well known, a member optimally designed for buckling load only, exhibits $w_{xx} \sim (1-x)^{(1-n)/(n+1)}$, $\alpha \sim (1-x)^{2/(n+1)}$ close to $x = 1$. In order to investigate the behaviour of $u(x)$ in the vicinity of $x = 1$ write

$$\alpha(x) = C(x)\,(1-x)^{2/(n+1)} \tag{30}$$

where $C(x)$ is a regular non-zero function, and expand $u(x)$ in the neighbourhood of $x = 1$ as

$$u(x) = 1 + D(1-x)^{\kappa} + \cdots \tag{31}$$

where κ is a positive non-integer number and D is constant. Substituting eqns (30) and (31) into the differential equation, eqn (19), and considering the lowest power of $(1-x)$ as well as the need to satisfy boundary condition (22), shows that $\kappa = (n-1)/(n+1)$.

Introducing a regular function $f(x)$, we put

$$u_x = f(x)\,(1-x)^{-2/(n+1)}. \tag{32}$$

Integrating the differential equation, eqn (19), once with respect to x and using boundary condition (22), we obtain

$$\alpha u_x = \beta\left[1 + \int_x^1 \alpha u \,\mathrm{d}\eta\right] = h(x). \tag{33}$$

Substituting eqns (30) and (32) into eqn (33) we get

$$f(x) = h(x)C(x). \tag{34}$$

The fundamental frequency of longitudinal vibrations of the optimally designed column was found as follows with [0, 1] divided into a large number of subintervals.

(i) Assume $f(x) \equiv 1$ in the first iteration.
(ii) Find

$$u(x) = \int_0^x f(\eta)(1-\eta)^{-2/(n+1)}\,d\eta.$$

(iii) Normalize $u(x)$ and $f(x)$ so that $u(1) = 1$.
(iv) Calculate

$$\beta = \frac{\int_0^1 \alpha u_x^2\,dx}{\int_0^1 \alpha u^2\,dx+1} = \frac{\int_0^1 C(x)f^2(x)(1-x)^{-2/(n+1)}\,dx}{\int_0^1 C(x)(1-x)^{2/(n+1)}u^2\,dx+1}.$$

(v) Calculate a new value for $f(x)$

$$f(x) = h(x)/C(x).$$

(vi) Repeat steps (ii)–(v) until convergence of successive values of β where $\alpha(x)$ and $C(x)$ refer to the optimal column and explicit expressions were formulated for integration, within each subinterval, of terms containing powers of $(1-x)$.

The results are shown in Figs 2–4 for $n = 1, 2, 3$, respectively. In region I the fun-

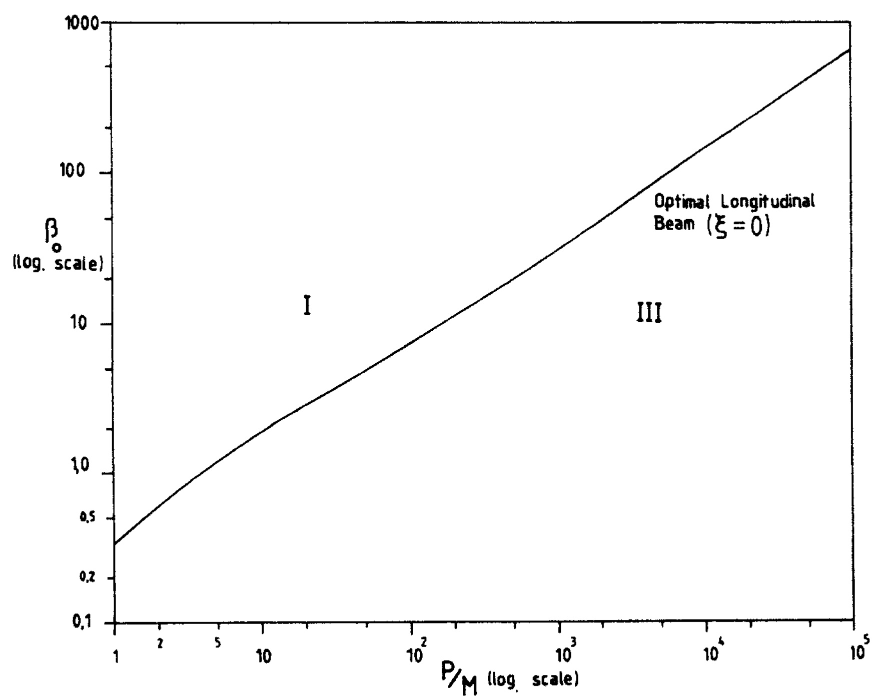

Fig. 2. Division of the design space; $n = 1$.

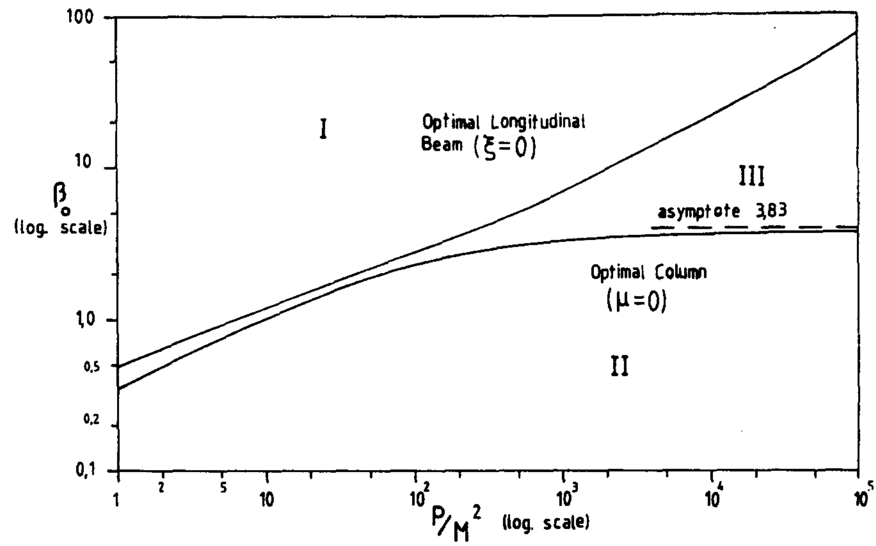

Fig. 3. Division of the design space; $n = 2$.

damental frequency of longitudinal vibration alone controls the design; in region II the buckling loads alone controls the design; and in region III both constraints are active.

It may be noted in the case of $n = 1$, that because of the nature of the singularity at the free end of the optimal column, longitudinal vibrations are not possible. This means that, for $n = 1$, region II contains only the P-axis (i.e. points where $\beta = 0$).

3.2. *Dual-constraint optimization*

Before discussing the solution in the case where both constraints are active, $\mu > 0$, $\xi > 0$, it is necessary to check whether singularities occur in the solution for this case. To that end we expand w, u and α about $x = 1$

$$\begin{aligned} w &= w(1)+A_1(1-x)+\cdots+B_1(1-x)^p+\cdots \\ u &= u(1)+\cdots+B_2(1-x)^q+\cdots \\ \alpha &= C(1-x)^r+\cdots \end{aligned} \tag{35}$$

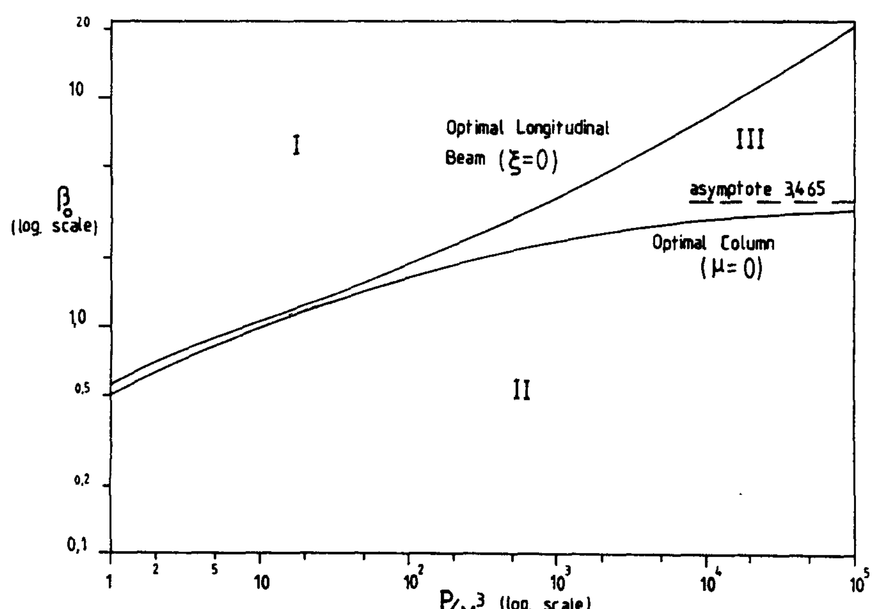

Fig. 4. Division of the design space; $n = 3$.

where p is the lowest power in the expansion of w, not 0 or 1, q is the lowest non-zero power in the expansion of u, and r is the lowest power in the expansion of α about $x = 1$.

Substituting eqns (35) into the differential equations, eqns (19) and (20), and optimality condition (26) and considering the lowest powers of $(1-x)$ shows that

$$p = 3, \qquad q = 1, \qquad r = 0. \tag{36}$$

Furthermore the form of the solution suggested by eqns (35) and (36) is capable of satisfying boundary condition (22) at $x = 1$. It is therefore concluded that singularities cannot occur in the solution at $x = 1$ when both constraints are active.

Attention is now turned to the solution of the problem with both constraints active.

Multiplying optimality condition (26) by α^{n+1}, making use of eqns (20) and (33), and rearranging gives

$$\alpha^{n+1} = \frac{\alpha^{n-1}\mu\beta_0^2\left(1+\int_x^1 \alpha u \, dx\right)^2 + \xi\left(\frac{P}{M^n}(1-w)\right)^2}{1+\mu\beta_0 u^2}. \tag{37}$$

The solution was again generated numerically at a large number of equally spaced discrete points on [0, 1].

The algorithm used is given below.

(i) Specify β_0, P_0/M^n.

(ii) Set up initial approximations for u_x, w_{xx} and $\alpha(x)$

$$u_x(x) = 1$$
$$w_{xx}(x) = 3(1-x)$$
$$\alpha(x) = \left(4\frac{P_0}{M^n}\pi^2\right)^{1/n}.$$

(iii) Estimate values for μ and ξ.

(iv) Find u_x, u, β for the assumed values of $\alpha(x)$:

(a) find

$$u(x) = \int_0^x u_\eta \, d\eta;$$

(b) normalize $u_x(x)$, $u(x)$ so that $u(1) = 1$;

(c) find

$$\beta = \frac{\int_0^1 \alpha u_x^2 \, dx}{1+\int_0^1 \alpha u^2 \, dx};$$

(d) find a new estimate for u_x

$$u_x(x) = \beta\left[1+\int_x^1 \alpha u \, dx\right]\Big/\alpha(x);$$

(e) repeat steps (iva–d) until consecutive values of $u_x(1)$ converge.

(v) Find w_{xx}, w_x, w and P_0/M^n for the assumed values of $\alpha(x)$:

(a) find

$$w_x = \int_0^x w_{\eta\eta}\,\mathrm{d}\eta$$

$$w = \int_0^x w_\eta\,\mathrm{d}\eta;$$

(b) normalize w_{xx}, w_x, w so that $w(1) = 1$;

(c) find the critical load

$$\frac{P}{M^n} = \frac{\int_0^1 \alpha^n w_{xx}^2\,\mathrm{d}x}{\int_0^1 w_x^2\,\mathrm{d}x};$$

(d) find a new estimate for $w_{xx}(x)$

$$w_{xx} = \frac{P}{M^n}(1-w)/\alpha^n;$$

(e) repeat steps (va–d) until consecutive values of $w_{xx}(1)$ converge.

(vi) Find a new approximation for $\alpha(x)$

$$\alpha_i = \alpha^r \cdot \alpha_{i-1}^{1-r}$$

where

$$\alpha = \left[\frac{\mu\beta^2\alpha^{n-1}\left(1+\int_x^1 \alpha u\,\mathrm{d}x\right)^2 + \xi\left(\frac{P}{M^n}(1-w)\right)^2}{1+\mu\beta u^2}\right]^{1/n+1}$$

and the subscript refers to the iteration number. The implicit equation for α was solved by repeated substitution for the α^{n-1} term—the integral was not re-evaluated at this step. Values of r between 0.4 and 0.7 were found suitable.

(vii) Store the current values of $u_x(1)$, $w_{xx}(1)$, β, P/M^n and repeat steps (iv)–(vi) until the consecutive values of these quantities converge.

(viii) Repeat steps (iii)–(vii) using Newton's method on μ and ξ until

$$\beta = \beta_0$$

$$\frac{P}{M^n} = \frac{P_0}{M^n}.$$

The gradients were found numerically so that each iteration of Newton's method required three evaluations of steps (iii)–(vii). At steps (ive) and (ve) the convergence criteria used was successive differences of less than 0.01% and at step (vii) 0.001% was found suitable.

4. DISCUSSION AND RESULTS

Steps (ii)–(vii) in the algorithm described above represent an "inverse" solution to the problem where the Lagrange parameters, μ, ξ are specified and the corresponding optimal

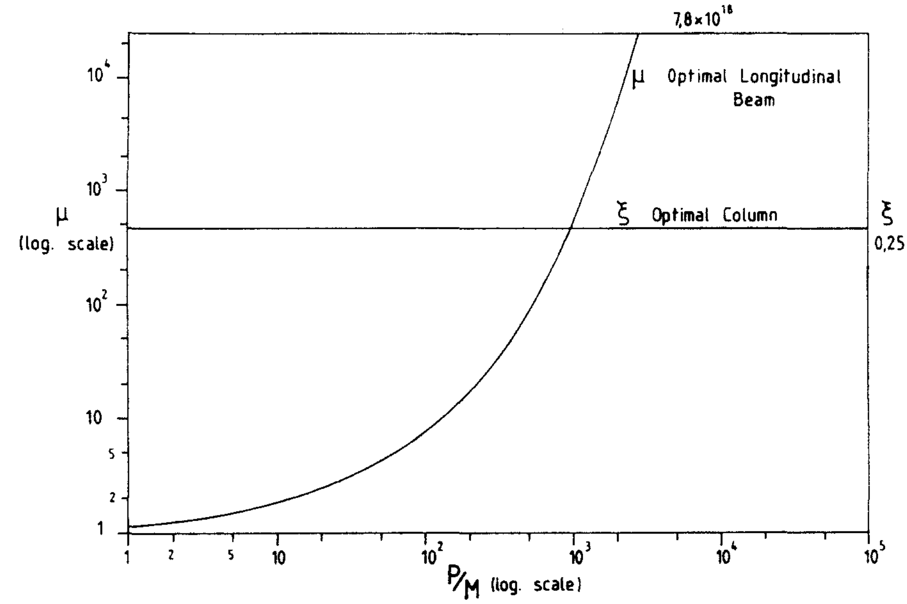

Fig. 5. Values of Lagrange multipliers μ, ξ for the limiting cases and $n = 1$.

design is found. In most cases convergence was achieved quickly, within about 20 iterations. In a few cases the application of Newton's method in step (viii) caused the values of μ, ξ to overshoot to infeasible values. The difficulty was overcome by restricting the amount by which values of μ and ξ were allowed to change on a single iteration.

The extreme values of the Lagrange multipliers against P/M^n are shown in Figs 5–7 for $n = 1$–3. These are useful in choosing the initial estimates of μ and ξ. The straight line relationships (on log–log scales) of ξ against P/M^n for the optimal column means that

$$\xi \propto P^{(1-n)/n}$$

for the optimal column, which is consistent with both $\alpha(x)$ and $u(x)$ having geometrically similar shapes (but different magnitudes) for different values of P.

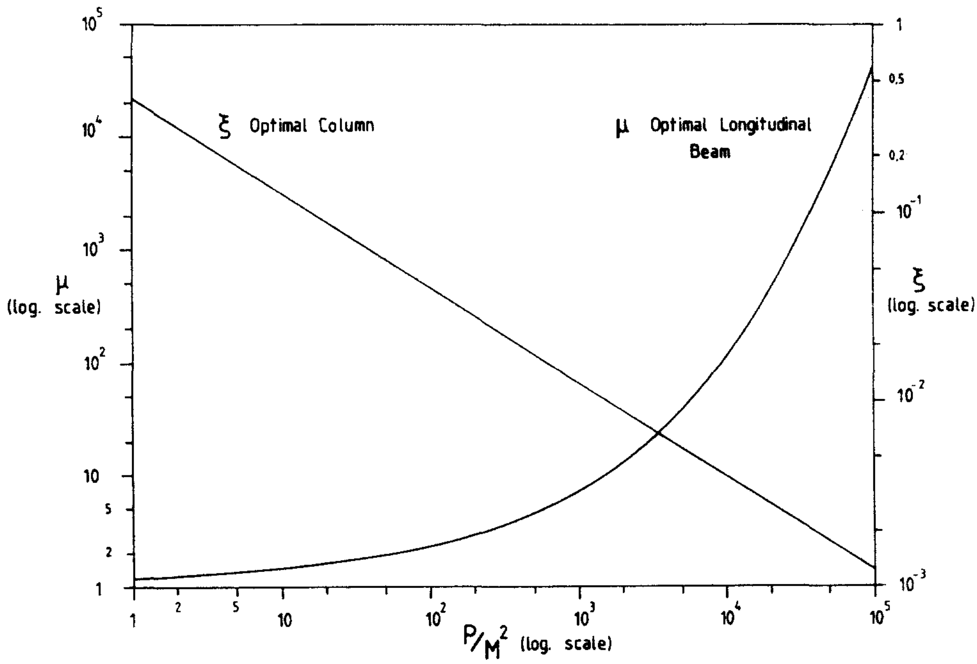

Fig. 6. Values of Lagrange multipliers μ, ξ for the limiting cases and $n = 2$.

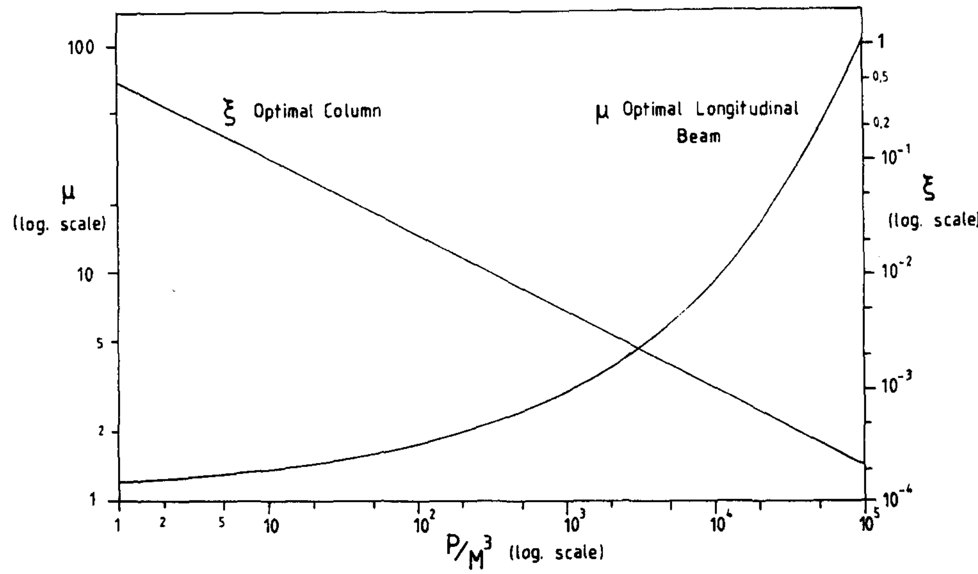

Fig. 7. Values of Lagrange multipliers μ, ξ for the limiting cases and $n = 3$.

The shapes of the optimal dual purpose beam are compared to those of the optimal single purpose design in Figs 8–10 and, in more detail, in Tables 1–9 for various values of P/M^n and β.

The optimal design may be compared with a prismatic beam in order to judge the effectiveness of the optimization. A prismatic beam satisfying the design constraints has a value of α given by the greater of

$$\alpha = \sqrt{\beta_0}\tan\sqrt{\beta_0} \tag{38}$$

or

$$\alpha = \left[\frac{4}{\pi^2}\left(\frac{P_0}{M^n}\right)\right]^{1/n}. \tag{39}$$

It may be noted from eqn (38) that a prismatic design is not possible for $\beta_0 \geqslant \pi^2/4$, whereas an optimal design is still possible. The savings are shown in Table 10 for various values of β, P/M^n.

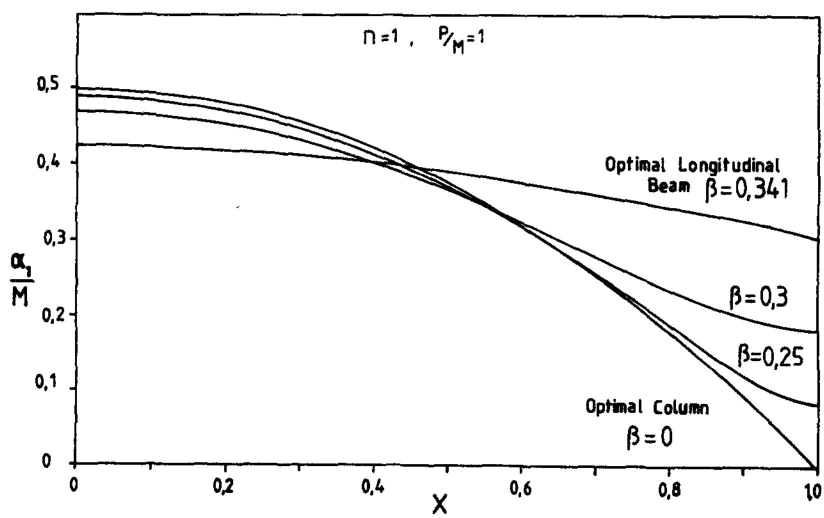

Fig. 8. Shape of optimal member compared with the optimal column and optimal longitudinally vibrating beam for $P/M = 1$, various values of β and $n = 1$.

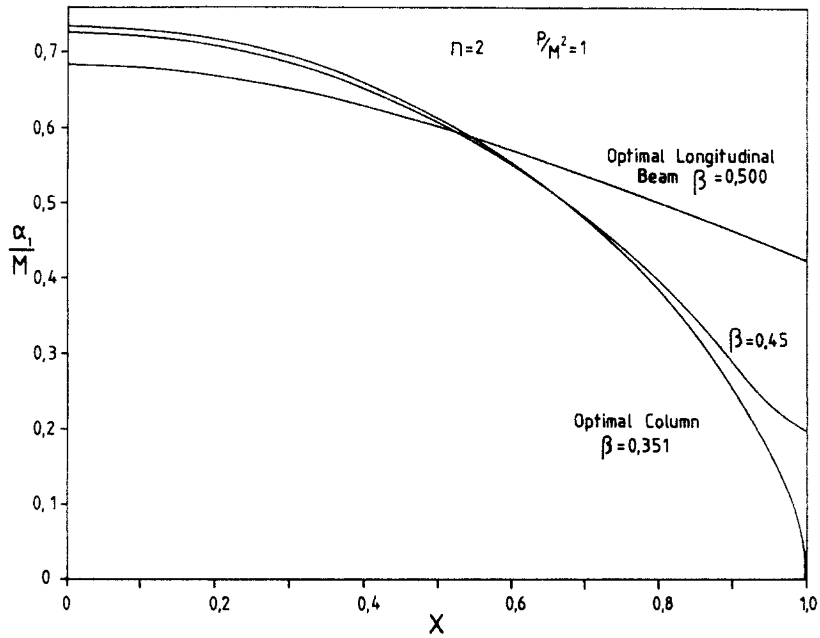

Fig. 9. Shape of optimal member compared with the optimal column and optimal longitudinally vibrating beam for $P/M^2 = 1$, $\beta = 0.45$, and $n = 2$.

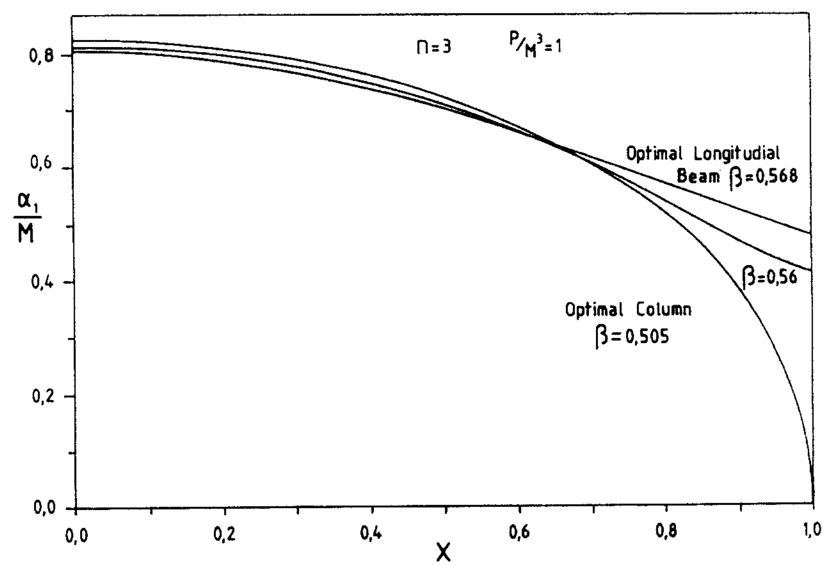

Fig. 10. Shape of optimal member compared with the optimal column and longitudinally vibrating beam for $P/M^3 = 1$, $\beta = 0.56$ and $n = 3$.

Table 1. $n = 1$, $P/M = 1$

	Optimum column $\beta = 0.00$	$\beta = 0.05$	$\beta = 0.10$	$\beta = 0.20$	$\beta = 0.30$	Optimum longitudinal beam $\beta = 0.341$
μ	0.0	0.2644×10^{-4}	0.8656×10^{-4}	0.01357	0.4236	1.119
ξ	0.25	0.24999	0.24999	0.2495	0.1725	0.0
x			α_1/M			
0.0	0.500	0.500	0.500	0.499	0.469	0.424
0.1	0.495	0.495	0.495	0.494	0.465	0.423
0.2	0.480	0.480	0.480	0.479	0.452	0.418
0.3	0.455	0.455	0.455	0.454	0.432	0.411
0.4	0.420	0.420	0.420	0.419	0.404	0.402
0.5	0.375	0.375	0.375	0.374	0.368	0.390
0.6	0.320	0.320	0.320	0.319	0.327	0.376
0.7	0.255	0.255	0.255	0.255	0.282	0.360
0.8	0.180	0.180	0.180	0.181	0.238	0.343
0.9	0.0950	0.0950	0.0950	0.0984	0.202	0.326
1.0	0.000	0.257×10^{-3}	0.930×10^{-3}	0.0298	0.184	0.307

Table 2. $n = 1$, $P/M = 10$

	Optimum column $\beta = 0.00$	$\beta = 0.05$	$\beta = 1.0$	$\beta = 1.5$	$\beta = 1.75$	Optimum longitudinal beam $\beta = 1.887$
μ	0.0	0.2712×10^{-4}	0.1137×10^{-3}	0.7800×10^{-2}	0.08956	1.810
ξ	0.25	0.24999	0.24999	0.24929	0.2397	0.0
x			α_1/M			
0.0	5.00	5.00	5.00	5.00	5.00	5.34
0.1	4.95	4.95	4.95	4.95	4.94	5.24
0.2	4.80	4.80	4.80	4.80	4.79	4.95
0.3	4.55	4.55	4.55	4.55	4.53	4.52
0.4	4.20	4.20	4.20	4.20	4.18	4.00
0.5	3.75	3.75	3.75	3.75	3.72	3.44
0.6	3.20	3.20	3.20	3.20	3.17	2.89
0.7	2.55	2.55	2.55	2.55	2.53	2.37
0.8	1.80	1.80	1.80	1.80	1.80	1.92
0.9	0.950	0.950	0.950	0.955	1.03	1.53
1.0	0.00	0.260×10^{-2}	0.0107	0.132	0.487	1.21

Table 3. $n = 1$, $P/M = 1000$

	Optimum column $\beta = 0.0$	$\beta = 5$	$\beta = 10$	$\beta = 20$	$\beta = 25$	Optimum longitudinal beam $\beta = 31.0$
μ	0.0	0.08227	1.858	27.62	104.4	556.6
ξ	0.25	0.2440	0.2764	0.7081	1.075	0.0
x			α_1/M			
0.0	500.0	543.0	1630.0	12,200.0	31,000.0	96,300.0
0.1	495.0	535.0	1490.0	10,100.0	24,000.0	71,600.0
0.2	480.0	509.0	1160.0	6030.0	13,000.0	33,800.0
0.3	455.0	470.0	809.0	2960.0	5610.0	12,700.0
0.4	420.0	420.0	525.0	1320.0	2200.0	4360.0
0.5	375.0	362.0	333.0	576.0	840.0	1450.0
0.6	320.0	299.0	211.0	254.0	321.0	479.0
0.7	255.0	232.0	135.0	117.0	127.0	158.0
0.8	180.0	161.0	82.9	56.7	52.9	51.7
0.9	95.0	84.0	41.1	24.8	20.8	17.0
1.0	0.0	1.21	3.08	4.47	5.00	5.57

Table 4. $n = 2$, $P/M^2 = 1$

	Optimum column $\beta = 0.3513$	$\beta = 0.375$	$\beta = 0.400$	$\beta = 0.450$	$\beta = 0.475$	Optimum longitudinal beam $\beta = 0.4998$
μ	0.0	0.2635×10^{-2}	0.01834	0.2168	0.5176	1.178
ξ	0.3972	0.3966	0.3930	0.3372	0.2401	0.0
x			α_1/M			
0.0	0.735	0.735	0.734	0.726	0.713	0.684
0.1	0.731	0.730	0.730	0.721	0.709	0.680
0.2	0.717	0.717	0.716	0.708	0.696	0.670
0.3	0.694	0.693	0.693	0.685	0.674	0.654
0.4	0.660	0.660	0.659	0.652	0.643	0.632
0.5	0.615	0.615	0.614	0.608	0.603	0.605
0.6	0.557	0.557	0.557	0.553	0.553	0.574
0.7	0.483	0.483	0.483	0.484	0.493	0.540
0.8	0.388	0.388	0.388	0.398	0.425	0.504
0.9	0.258	0.258	0.260	0.292	0.354	0.468
1.0	0.00	0.0192	0.0540	0.200	0.306	0.430

Table 5. $n = 2$, $P/M^2 = 10$

	Optimum column $\beta = 1.00$	$\beta = 1.05$	$\beta = 1.10$	$\beta = 1.15$	$\beta = 1.20$	Optimum longitudinal beam $\beta = 1.210$
μ	0.0	0.002975	0.1140	0.1165	0.8021	1.474
ξ	0.1256	0.1254	0.1242	0.1140	0.05824	0.0
x			α_1/M			
0.0	2.32	2.32	2.32	2.33	2.37	2.45
0.1	2.31	2.31	2.31	2.31	2.35	2.42
0.2	2.27	2.27	2.27	2.27	2.29	2.34
0.3	2.19	2.19	2.19	2.19	2.19	2.20
0.4	2.09	2.09	2.09	2.08	2.05	2.03
0.5	1.95	1.94	1.94	1.94	1.89	1.83
0.6	1.76	1.76	1.76	1.75	1.69	1.63
0.7	1.53	1.53	1.52	1.52	1.47	1.42
0.8	1.23	1.23	1.23	1.22	1.22	1.23
0.9	0.817	0.817	0.819	0.835	0.965	1.04
1.0	0.00	0.0572	0.154	0.365	0.767	0.880

Table 6. $n = 2$, $P = 1000$

	Optimum column $\beta = 3.27$	$\beta = 3.5$	$\beta = 4.5$	$\beta = 5.5$	$\beta = 6.5$	Optimum longitudinal beam $\beta = 6.83$
μ	0.0	0.1955	1.336	2.982	5.568	6.754
ξ	0.01256	0.01142	0.7195×10^{-2}	0.4296×10^{-2}	0.1253×10^{-2}	0.00
x			α_1/M			
0.0	23.2	25.2	40.1	65.0	104.0	122.0
0.1	23.1	24.9	38.6	61.7	97.8	114.0
0.2	22.7	24.1	34.7	52.8	81.3	93.9
0.3	21.9	22.7	29.4	41.7	61.1	69.6
0.4	20.9	21.1	23.9	30.9	42.6	47.7
0.5	19.5	19.1	19.1	22.1	28.2	31.0
0.6	17.6	16.8	15.1	15.7	18.2	19.5
0.7	15.3	14.3	11.8	11.2	11.5	11.9
0.8	12.3	11.3	8.90	7.88	7.32	7.22
0.9	8.17	7.42	5.80	5.04	4.50	4.33
1.0	0.00	1.19	1.96	2.28	2.52	2.59

Table 7. $n = 3$, $P/M^3 = 1$

	Optimum column $\beta = 0.505$	$\beta = 0.52$	$\beta = 0.54$	$\beta = 0.56$	Optimum longitudinal beam $\beta = 0.568$
μ	0.00	0.04978	0.2504	0.7931	1.204
ξ	0.4642	0.4483	0.3781	0.1691	0.00
x			α_1/M		
0.0	0.825	0.825	0.821	0.814	0.808
0.1	0.821	0.821	0.818	0.810	0.804
0.2	0.810	0.810	0.806	0.806	0.790
0.3	0.792	0.790	0.786	0.797	0.768
0.4	0.764	0.762	0.753	0.748	0.739
0.5	0.726	0.725	0.720	0.710	0.704
0.6	0.676	0.675	0.671	0.664	0.663
0.7	0.610	0.609	0.607	0.608	0.619
0.8	0.520	0.520	0.522	0.543	0.573
0.9	0.386	0.389	0.404	0.472	0.527
1.0	0.00	0.114	0.254	0.415	0.480

Table 8. $n = 3$, $P/M^3 = 10$

	Optimum column $\beta = 0.966$	$\beta = 0.985$	$\beta = 1.00$	$\beta = 1.02$	$\beta = 1.03$	Optimum longitudinal beam $\beta = 1.04$
μ	0.0	0.02966	0.1057	0.4402	0.8959	1.398
ξ	0.1000	0.09807	0.09283	0.06879	0.03559	0.0
x			α_1/M			
0.0	1.78	1.78	1.78	1.80	1.84	1.92
0.1	1.77	1.77	1.77	1.79	1.83	1.90
0.2	1.75	1.75	1.75	1.76	1.79	1.85
0.3	1.71	1.71	1.71	1.71	1.72	1.75
0.4	1.65	1.64	1.64	1.64	1.64	1.64
0.5	1.56	1.56	1.56	1.55	1.53	1.50
0.6	1.46	1.45	1.45	1.43	1.40	1.35
0.7	1.31	1.31	1.31	1.28	1.25	1.20
0.8	1.12	1.12	1.11	1.10	1.08	1.05
0.9	0.833	0.833	0.835	0.851	0.881	0.913
1.0	0.00	0.167	0.309	0.562	0.703	0.784

Table 9. $n = 3$, $P/M^3 = 1000$

	Optimum column $\beta = 2.39$	$\beta = 2.75$	$\beta = 3.00$	$\beta = 3.25$	$\beta = 3.5$	Optimum longitudinal beam $\beta = 3.67$
μ	00.00	1.008	1.584	2.116	2.644	3.006
ξ	0.4642×10^{-2}	0.2192×10^{-2}	0.1264×10^{-2}	0.6700×10^{-3}	0.2374×10^{-3}	0.00
x			α_1/M			
0.0	8.25	10.9	13.4	16.3	19.6	22.1
0.1	8.22	10.7	13.0	15.8	18.9	21.3
0.2	8.10	10.0	12.0	14.4	17.1	19.1
0.3	7.92	9.17	10.6	12.4	14.5	16.1
0.4	7.64	8.17	9.02	10.2	11.7	12.9
0.5	7.26	7.18	7.52	8.19	9.12	9.87
0.6	6.76	6.23	6.21	6.44	6.90	7.32
0.7	6.10	5.23	5.11	5.05	5.14	5.29
0.8	5.20	4.38	4.11	3.93	3.81	3.76
0.9	3.86	3.23	3.02	2.87	2.73	2.64
1.0	0.00	1.42	1.57	1.68	1.78	1.83

Table 10. Comparison of the optimally designed beam with the prismatic design

	$n = 1$	
P/M	β	$\dfrac{V_{prism} - V_{opt}}{V_{prism}} \times 100\%$
1	0.10	17.8
10	1.0	17.8
1000	10.0	†
	$n = 2$	
P/M^2	β	$\dfrac{V_{prism} - V_{opt}}{V_{prism}} \times 100\%$
1	0.4	13.4
10	1.10	13.4
1000	4.5	†
	$n = 3$	
P/M^3	β	$\dfrac{V_{prism} - V_{opt}}{V_{prism}} \times 100\%$
1	0.54	10.7
10	1.02	11.0
1000	3.00	†

† Denotes that a prismatic design is not possible.

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